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Shoaling of finite-amplitude surface waves on water of slowly-varying depth

By M. STIASSNIE† AND D. H. PEREGRINE
Department of Mathematics, University of Bristol, England

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Periodic wave trains propagating over water which varies in depth in the direction of wave propagation are studied by using accurate solutions for wave trains in constant depth of water. The accurate solutions are (i) Cokelet's (1977) extension of Stokes' approximation and, for the longer waves, (ii) a solution for trains of solitary waves using the solitary-wave solution of Longuet-Higgins & Fenton (1974).

A representative selection of results is shown in diagrams. A feature which arises from the use of these accurate solutions is that near the highest wave two solutions are possible for a given incoming wave. Although the solutions cannot describe waves that break, it is shown that as depth is decreased a point is reached beyond which no solution can be found. This is taken to indicate the region in which waves break.

The limitations of the theory are discussed and analysed. Comparisons with experimental measurements of Hansen & Svendsen (1979) are included.

CONTENTS

1. Introduction

One reasonably successful and much used method of describing the behaviour of water waves propagating on water of varying depth is to assume that the depth variation has a sufficiently long scale that at any point the waves are described to a sufficient approximation by a uniform wave-train on a constant depth of water. In the simplest case, where the waves are so gentle that linear theory is adequate, the propagation paths, or rays, of waves can be calculated over relatively complex

† Permanent address: Faculty of Civil Engineering, Technion-Israel Institute of Technology, Haifa, Israel.

topography. However, in the commonest application, for waves approaching a coast, as the depth of water decreases so the wave steepness increases and linear theory becomes inaccurate. On beaches, waves usually steepen until they break.

If finite-amplitude solutions for periodic waves are used the calculation of propagation paths in a simple manner is no longer possible. When the governing equations are analysed (see Whitham 1967) it is found that there are four distinct characteristics of the equations indicating that in general some wave properties propagate in different directions. Also for moderate to deep water the characteristic directions are complex indicating an elliptic character for the equations. We do not enter this problem here, since we only consider the relatively simple problem where the bottom topography varies in one direction only. We simplify further by only considering waves propagating in this direction. (There is no intrinsic difficulty in considering the more general case of waves at an angle to such cylindrical geometry. This case is being investigated.) A typical example is a straight coastline with depth contours all parallel to the same direction. Examples of solutions to this problem are those of Svendsen & Brink-Kjær (1972) for cnoidal waves and James (1974*a*,*b*) who uses hyperbolic waves near the shore and Stokes' waves further out. The accurate deep-water-wave solution of Longuet-Higgins (1975) is used by Peregrine & Thomas (1979) to study the interaction of waves and currents.

Accurate results are given by Cokelet (1977) for steady gravity waves on any depth of water and of any steepness up to the maximum. This paper results from using these accurate solutions to describe waves normally incident on a beach. The description is in terms of waves propagating from deep water into a region where the depth monotonically decreases; but with the type of slowly-varying approximation used here, the actual variation of the depth is immaterial as long as it is sufficiently gentle.

The equations used to describe the waves assume very long scales both for wave modulation and for depth modulation. They are averaged equations and are derived in Stiassnie & Peregrine (1979) which develops work by Phillips (1966), Whitham (1967, 1974) and Crapper (1979) amongst others. The next section gives the equations in the simplified form needed here and also describes how Cokelet's solution method is used. In practice the accuracy of the solution becomes poor as the wavelength to depth ratio increases. It was found that a more satisfactory solution for long waves is a train of solitary waves (abbreviated to TSW) based on the accurate solitary-wave solution given by Longuet-Higgins & Fenton (1974). This TSW approximation has the unusual and useful property that it improves as the wave height increases. It is described in § 3.

Results of amplitude variation for a full range of deep-water wave steepnesses are described in § 4, together with selected examples of other wave properties. Comparison is made with linear theory and the cnoidal wave solution of Svendsen & Brink-Kjær (1972). The accurate wave solutions show maxima of all integral wave properties for waves slightly less than the steepest. This leads to the existence of two solutions of the equations for waves near the steepest. For given deep-water conditions there is a minimum depth at which a solution can be found and the corresponding wave although steep is not the steepest for that depth.

A theoretical solution based on averaging symmetrical wave trains, such as the one used here, cannot predict wave breaking. However, it is reasonable to expect breaking in the neighbourhood of a point where solutions cease to exist, and that the amplitude at breaking may well be close to the amplitude of that limiting solution. In this respect the accurate wave solutions are a marked advance on other solutions which rely on the specification of distinct breaking criteria. Section 5 includes a discussion of wave breaking in relation to these solutions and the properties of the limiting solutions are presented as functions of deep-water wave steepness.

The limitations of the theory are discussed and analysed in §§ 6 and 7. In particular, an analysis of the rate of interaction between two equal solitary waves indicates that once waves reach the stage of being described by the TSW approximation then it is only for extremely gentle slopes, less than 10^{-3}, that this theory can be appropriate. In most circumstances, namely less gentle slopes, once crests of waves become similar to solitary waves then an approach such as that of Miles (1979) is more suitable. However, in practice our results will frequently only differ a little from such a solution.

There are difficulties in making satisfactory comparisons with experiments. These are due to the dissipation, reflexion and secondary flows which occur in experiments, and to experimental difficulties of generating a wave train free from extraneous harmonics. Comparisons are given in § 8 with measurements of Hansen & Svendsen (1979). These show the importance of dissipation and difficulties of obtaining precise experimental results even when great care is taken over wave generation. However, encouraging and reasonable agreement is obtained, even in the vicinity of the highest waves.

This paper does not present the most efficient method for calculating particular solutions, nor does it attempt an exhaustive presentation of results that could be consulted for any conditions. However, a person involved in studying particular examples will find some guidance.

2. Theory and method of solution

A monochromatic plane wave train approaching a beach over gentle variations of depth with depth contours parallel to straight wave crests is considered. If reflexion and dissipation are neglected then four quantities are constant throughout the shoaling region until a point is reached where waves break. They are wave frequency ω^*, wave-action flux B^*, a Bernoulli constant γ^*, and the mass flux q^*. (An asterisk is used to denote dimensional quantities.) If these four quantities are determined by a wave train and water conditions at one point, then in principle the wave and water conditions at another point where the depth differs can be found if the waves remain as a periodic wave train.

The depth must be specified relative to a horizontal reference level, which cannot be the mean water level since that varies from place to place. The actual reference level is arbitrary; for convenience we choose the mean water level in deep water and refer to the corresponding depth (distance to the bottom), h^*, as the undisturbed depth of water.

The mean current is directly related to the reference frame chosen for defining the wave-train solution. We follow Cokelet (1977) and many others back to Stokes (1847) in choosing the reference frame in which the average flow at any point below a wave trough is zero. This means there is a mass-transport associated with the waves. In this work the total mass flow is taken equal to zero as would be the case for an impervious beach.

The mean water depth, D^*, and the current, U^*, are two of the four parameters required to determine the water and wave conditions at one point. Two other, purely-wave, parameters are required. In the examples which we calculate we use frequency and deep-water steepness. This latter quantity can be specified in various ways, for example, as H^*/L^* where H^* is wave height and L^* is wavelength, as a^*k^* where $a^* = \frac{1}{2}H^*$ is wave amplitude and $k^* = 2\pi/L^*$ is wavenumber, or implicitly in several ways. For several cases we use the expansion parameter $\epsilon^2 = 1 - q^2_{\text{crest}}\, q^2_{\text{trough}}/c^4$, used by Cokelet (1977) since this enables all the deep-water wave properties to be readily determined from his tables. The parameter $\epsilon^2 = 0$ for infinitesimal waves and increases monotonically to $\epsilon^2 = 1$ for the steepest wave.

With the results of Crapper (1979) and the further discussion of Stiassnie & Peregrine (1979) it is a simple matter to write the four constants in terms of a current U^*, the mean depth of water D^*, and properties of the wave train. This gives

$$\omega^* = k^*(c^* + U^*), \tag{1}$$

$$B^* = [U^*I^* + 3T^* - 2V^* + \tfrac{1}{2}\rho D^*\overline{(u_b^*)^2}]/k^*, \tag{2}$$

$$\gamma^* = g(D^* - h^*) + \tfrac{1}{2}U^{*2} + \tfrac{1}{2}\overline{(u_b^*)^2} \tag{3}$$

and

$$q^* = \rho U^* D^* + I^*. \tag{4}$$

The wave properties are the phase velocity c^*, the wave momentum density I^*, the kinetic and potential energy densities T^* and V^*, and the mean of the square of the wave-induced water velocity at the bed $\overline{(u_b^*)^2}$.

The equations (1) to (4) are made dimensionless in the following manner. All the wave properties which vary with position are made dimensionless with appropriate combinations of ρ, g, k^*. This means that they correspond exactly with Cokelet's (1977) dimensionless variables. The constants on the left-hand sides of equations (1) to (4) and k^* are made dimensionless with ρ, g and the wavenumber of the wave train in deep water k^*_∞. For some applications it may be more convenient to use the frequency ω^*_∞ rather than k^*_∞. If the deep-water steepness is known then this is a matter of simple algebra, and if numerical results are sought the approximation to the dispersion relation suggested by Peregrine & Thomas (1979) may be useful. An advantage of using k^*_∞ is that the corresponding deep-water steepness of waves is immediately apparent from their dimensionless amplitude $a_\infty = a^*_\infty k^*_\infty$. In presenting results k^*_∞ is used for scaling all variables.

The resulting dimensionless form of equations (1) to (4) is

$$\omega = \omega^*/(gk^*_\infty)^{\frac{1}{2}} = k^{\frac{1}{2}}(c + U), \tag{5}$$

$$B = B^*(k^*_\infty)^3/\rho g = (UI + 3T - 2V + \tfrac{1}{2}D\overline{u_b^2})/k^3, \tag{6}$$

$$\gamma = \gamma^* k^*_\infty/g = (D - h + \tfrac{1}{2}U^2 + \tfrac{1}{2}\overline{u_b^2})/k, \tag{7}$$

$$q = q^*(k^*_\infty)^{\frac{3}{2}}/\rho g^{\frac{1}{2}} = (DU + I)/k^{\frac{3}{2}}. \tag{8}$$

Every wave quantity on the right-hand side of these equations except for k is a function of two parameters. Cokelet (1977) uses ϵ^2 and d as these parameters. The first of these has already been mentioned; d is a (depth) × (wavenumber) parameter which is the dimensionless depth of a uniform stream of velocity c which has the same mass flux as occurs beneath the wave train in a reference frame in which the wave surface is steady. For infinitesimal waves $d = D$.

In the example under consideration the mass flow is zero, so that equation (8) gives

$$U = -I/D, \tag{9}$$

and equation (5) then gives

$$k = \omega^2 D^2/(cD-I)^2. \tag{10}$$

These may be used to eliminate U and k from equations (6) and (7), yielding

$$(c-I/D)^6(-I^2/D+3T-2V+\tfrac{1}{2}D\overline{u_b^2}) = B\omega^6, \tag{11}$$

and

$$(c-I/D)^2(D-h+\tfrac{1}{2}\overline{u_b^2}+\tfrac{1}{2}I^2/D^2) = \gamma\omega^2 \tag{12}$$

respectively. The constants on the right-hand sides of equations (11) and (12) can be found at any reference point by determining the left-hand side at that point. For example in deep water we may write

$$B\omega^6 = c_\infty^6(3T_\infty - 2V_\infty) \tag{13}$$

and

$$\gamma\omega^2 = c_\infty^3(D_\infty - h_\infty) \tag{14}$$

where in this latter term $D_\infty - h_\infty$ is zero for our chosen reference level.

We follow Cokelet (1977) in the method of solving for the wave properties and hence can consider the two equations (11) and (12) as two equations for ϵ^2 and d once the constants and h are given. However, since the dependence of wave properties on d is very complicated, we solve the equations by first choosing d; equation (11) is then an equation to be solved for ϵ^2. Then h, k and U are found from equations (12), (10) and (9) respectively. All wave properties can be found from ϵ^2, d and k.

In Cokelet's (1977) method for finding wave properties a power series in ϵ^2 is found for each variable. The coefficients of a number of terms are evaluated, $N = 50$ is often appropriate, and the corresponding $(\frac{1}{2}N, \frac{1}{2}N)$ Padé approximant is used to evaluate the property. In the spirit of this approach, equation (11) is expressed as a finite power series in ϵ^2 for which zeros are to be determined. The equation was solved using a method based on Padé approximants given in Baker (1975, p. 80). The method gives an indication of how accurate the resulting solutions are. Once ϵ^2 is known all other quantities are also expressed as power series in ϵ^2 and evaluated by using Padé approximants. Double precision arithmetic was used on an ICL 2980 computer (16 decimal places).

It was found that as values of h (or d) were decreased the above approach fails to give accurate results. That is, only one, or no, significant digit could be found by the root-solving method. This occurred for values of d ranging from 0·63 for the higher waves to lower values. An indication of the limits of solution can be found in figure 3(*b*). More terms in the series, quadruple precision and different root-solving methods failed to improve matters. Examination of the tables in Cokelet (1977) also shows a similar, but smaller, loss of accuracy once e^{-d} is greater than 0·5 (i.e. $d < 0{\cdot}69$). As a result a different approach was used for the longer waves, and is described in the next section.

3. A train of solitary waves

Long periodic waves of moderate amplitude are described well by the near-linear cnoidal wave solution. As the parameter in the cn function approaches one the wave profiles look like a sequence of solitary waves with a flat uniform stretch of water

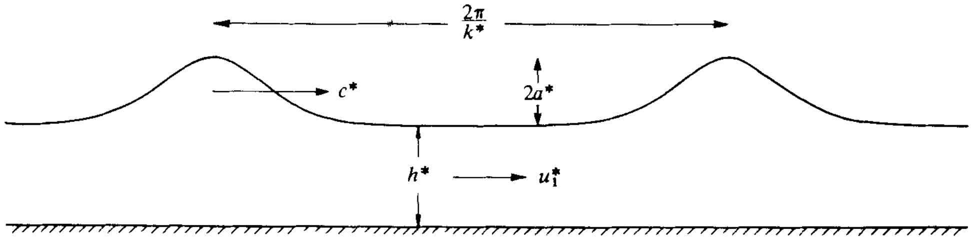


FIGURE 1. Definition diagram for a train of solitary waves.

between each crest. This aspect of water waves has long been noted (e.g. Munk 1949; Iwagaki 1968) and also applied to waves on a beach. Accurate integral properties of solitary waves are given by Longuet-Higgins & Fenton (1974). These are used to obtain accurate integral properties of a train of solitary waves (a TSW approximation). As is demonstrated in figures 2 and 3 this approximation agrees well with Cokelet's method of solution in a region where both methods can be expected to be accurate.

Longuet-Higgins & Fenton (1974) use an expansion parameter ω, which we denote by ω_s, and indicate that a [7, 7] Padé approximant is sufficient for 4 figure accuracy of integral properties for all waves up to the highest. Appropriate coefficients are given in that paper. The approximation is to take a solitary wave to represent each crest and for the interval between the outskirts of each crest to be a uniform flow, that is, water of constant depth and velocity. This can be done as a formal matching process to a higher order, using linear theory with cosh variation of the surface between each crest but that is not needed in the present study.

The waves are considered in two reference frames. The first corresponds to that of Cokelet's Stokes' wave solution, with zero mean flow below the level of the troughs. This means there must be a non-zero velocity, u_1^*, beneath the flat region of the wave, shown in figure 1. The depth of water in the flat region is h_s^*. The wavelength and height of the wave train are $2\pi/k^*$ and $2a^*$ and it has velocity c^*, in terms of Cokelet's notation. For the solitary wave we use Longuet-Higgins & Fenton's notation with the addition of a subscript s. Thus C_s is the total circulation of the solitary wave, ϵ_s its height, M_s its 'excess' mass and F_s its velocity. The quantity $k^*h_s^*$ appears frequently and is denoted by μ.

Consideration of the mean circulation per unit length, gives

$$C = \frac{1}{\lambda}\int_0^\lambda u\,dx = C^*(k^*/g)^{\frac{1}{2}} = \mu^{\frac{3}{2}}C_s/2\pi + u_1^*(k^*/g)^{\frac{1}{2}},$$

but in this first reference frame, $C = 0$ so that

$$u_1^* = -\frac{\mu^{\frac{3}{2}}C_s}{2\pi}\left(\frac{g}{k^*}\right)^{\frac{1}{2}}. \tag{15}$$

Thus
$$c = c^*(k^*/g)^{\frac{1}{2}} = \mu^{\frac{1}{2}}(F_s - \mu C_s/2\pi). \tag{16}$$

The mean depth
$$D = k^*D^* = \mu(1 + \mu M_s/2\pi). \tag{17}$$

Other quantities are most readily found by consideration of the waves in a second frame of reference in which the wave profile is at rest. Then the mass flow

$$Q = cd = Q^*k^{*\frac{3}{2}}/\rho g^{\frac{1}{2}} = \mu^{\frac{3}{2}}F_s, \tag{18}$$

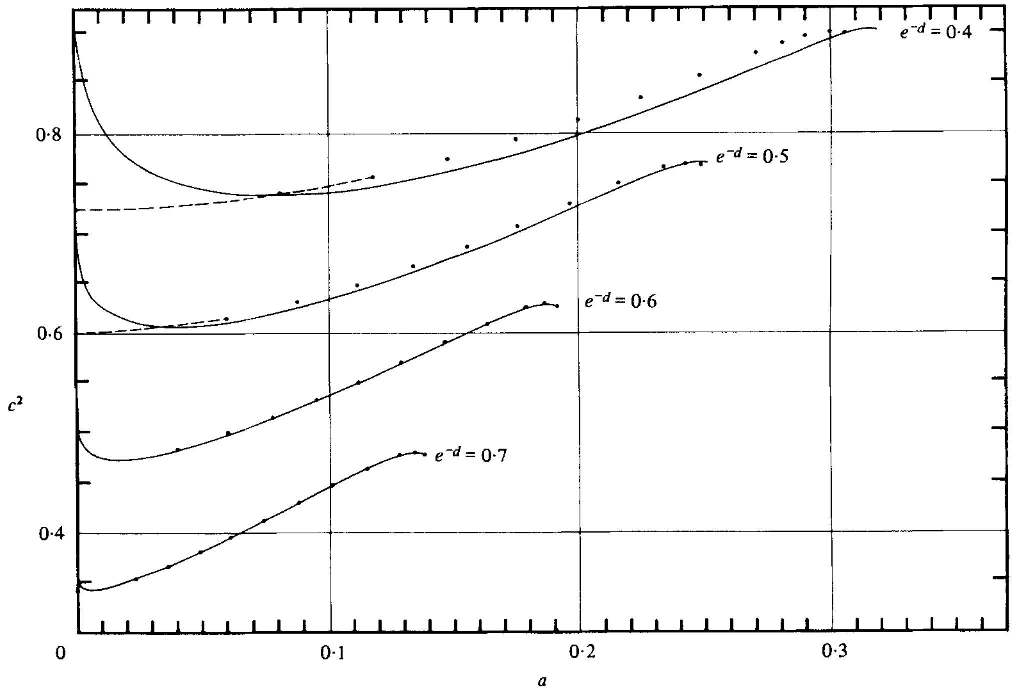


FIGURE 2. Comparison of phase velocity squared, c^2, against amplitude, a, for waves on finite depth of water between Cokelet's (1977) solution (--- ● ● ●) and the train of solitary waves approximation (——).

giving d, the total head

$$R = k^*R^* = \mu(1+\tfrac{1}{2}F_s^2), \tag{19}$$

and the momentum flux

$$S = S^*k^{*2}/\rho g = \mu^2(\tfrac{1}{2}+F_s^2), \tag{20}$$

are all readily determined in the uniform flow between the crests.

The above relations, (16) to (20), together with the relations given by Cokelet (1977) are sufficient to express the integral quantities in equations (5) to (8), (11) and (12) in terms of solitary wave properties and μ. The solitary-wave properties are all functions of the expansion parameter ω_s so that instead of equation (11) relating ϵ^2 and d, an equation involving μ and ω_s arises. Considered as a function of μ the equation is an eighth-order polynomial. Hence, given ω_s the solitary-wave properties are evaluated and μ is then found by standard root solving routines. All the other wave and water properties are then readily found.

Naturally, before using this TSW approximation it is desirable to check its accuracy. The simplest way to do this is to compare results calculated by the TSW approximation with the results tabulated by Cokelet (1977). A comparison of the two solution methods is given for c^2 as a function of a in figure 2.

The length scale of a solitary wave increases as its amplitude decreases, so the TSW approximation must fail as $a \to 0$. This failure is evident in the figure. Conversely as the amplitude increases the length-scale of a solitary wave decreases and the TSW approximation can be expected to improve. (A pleasant change from linear and near-linear solutions.) The difference between the solutions is significant for the whole range of amplitudes at $e^{-d} = 0{\cdot}5$ which corresponds to μ in the range 0·6 to 0·67, but the difference becomes very slight for lesser depths. Indeed for the higher waves, and $e^{-d} \geqslant 0{\cdot}7$, which is for $\mu \leqslant 0{\cdot}35$ we suspect that the TSW approximation is more accurate than Cokelet's solution since he only gives 3 or less significant figures for many properties. TSW is essentially a long wave approximation which implies $\mu = k^* h_s^* \ll 1$, so that it is gratifying to see the good agreement obtained for only moderately small values of μ.

4. Theoretical results

To illustrate the theory twelve different wave steepnesses in deep water were chosen and the corresponding wave trains in shallower water have been calculated up to the highest wave consistent with the imposed conditions. For presenting the results the single length scale $(k_\infty^*)^{-1}$ has been used to form dimensionless quantities so that the corresponding quantities of § 2 are multiplied by the appropriate power of k.

The chosen deep-water amplitudes are

$$a_\infty = a_\infty^* k_\infty^* = \begin{cases} 0{\cdot}371,\ 0{\cdot}307,\ 0{\cdot}232,\ 0{\cdot}187,\ 0{\cdot}131, \\ 0{\cdot}0918,\ 0{\cdot}0579,\ 0{\cdot}0409,\ 0{\cdot}0157, \\ 0{\cdot}314 \times 10^{-2},\ 0{\cdot}157 \times 10^{-2},\ 0{\cdot}336 \times 10^{-3}. \end{cases}$$

The first eight values correspond to $\epsilon_\infty^2 = 0{\cdot}7$, 0·5, 0·3, 0·2, 0·1, 0·05, 0·02 and 0·01. The remaining, smallest four values correspond to $H_\infty/L_\infty = 5 \times 10^{-3}$, 1×10^{-3}, 5×10^{-4} and $1{\cdot}07 \times 10^{-4}$ and were chosen to ease comparison with the cnoidal-wave results given in tabular form by Svendsen & Brink-Kjær (1972) and Svendsen & Hansen (1977). (Note, in the latter paper deep-water conditions should be incorporated by energy-flux conservation which means using both tables 1 and 2 of the paper.) For each of these values of a_∞ a considerable amount of results could be presented. To keep within reasonable bounds, all results for the wave amplitude are given, some more results are given only for $\epsilon_\infty^2 = 0{\cdot}7$ and 0·01, that is $a_\infty = 0{\cdot}371$ and 0·0409. Then, in the next section, wave breaking is discussed and results relevant to wave breaking are presented.

In figure 3 the results are given for amplitude as a function of the local undisturbed depth of water, $h = h^* k_\infty^*$. Three 'nested' diagrams are used in order that results for the steepest waves can be shown clearly. The results of four theoretical solutions are given in order that comparisons can be made. They are solutions using (i) Cokelet's method, (ii) the TSW approximation, (iii) cnoidal waves and (iv) linear theory. The individual points calculated are plotted and in the case where the Padé-approximant root-finding method indicated low accuracy (i.e. less than three significant figures) the points are marked.

Every solution curve which involves accurate steep-wave solutions (i.e. Cokelet's method and TSW) has a vertical tangent and is double valued for the highest waves. This is not surprising because of the maximum of all integral properties of waves just below the steepest waves. This feature also appears in the results of Peregrine & Thomas (1979) for steep waves on currents. For brevity we refer to this portion of the curve as the 'breaking region'. Whether or not such a description is appropriate, and the problems of interpreting this part of the solution are discussed in the next section.

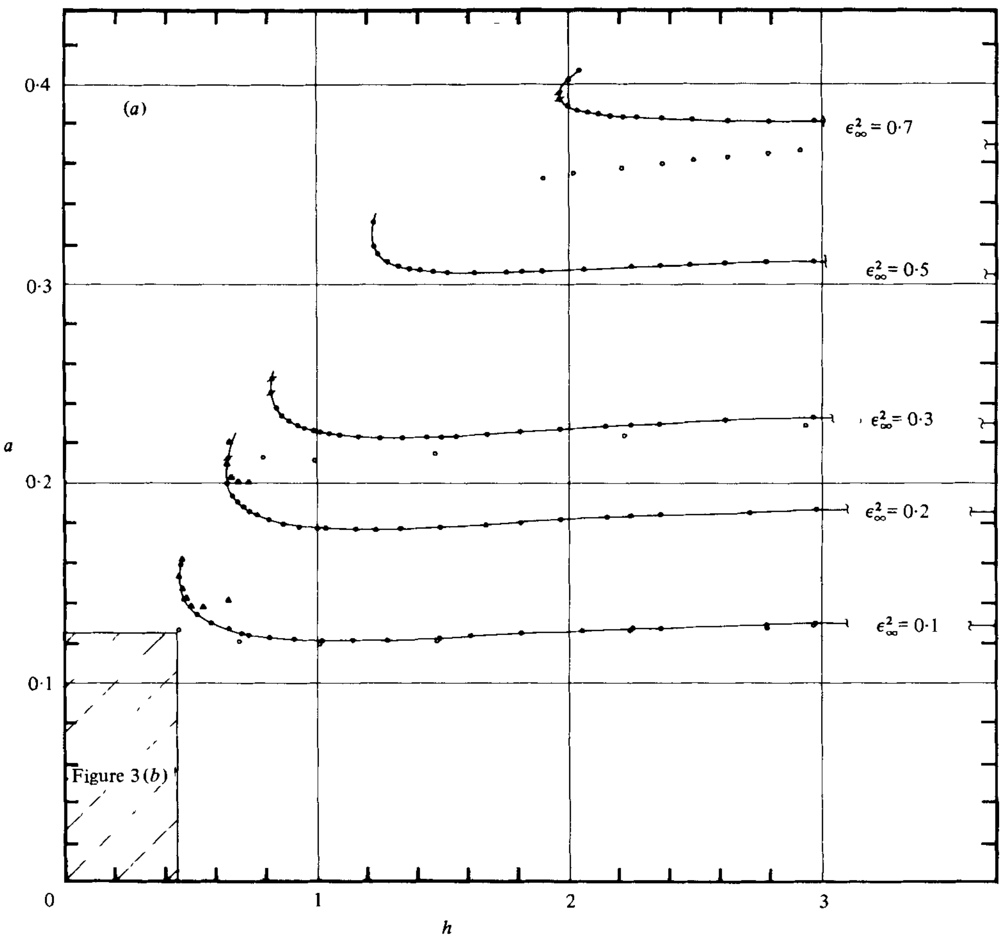
For waves which are already steep in deep water, say $\epsilon_\infty^2 \geqslant 0{\cdot}3$, it can be seen in figure 3(*a*) that there is little variation in amplitude until the final rapid rise to the breaking region. Unlike linear theory which predicts an initial decrease, this finite-amplitude solution shows a small initial increase in amplitude. For the steepest waves there is no diminution before the breaking region. Slightly lower waves, such as $\epsilon_\infty^2 = 0{\cdot}5$ show a rise and then a fall in amplitude. This relatively complicated behaviour is difficult to explain. Some of the problems of interpretation can be seen by examining Cokelet's (1977) figures 16, 14 and 15 for the flux of mass, momentum and energy respectively. Note the way in which his solution curves intersect for different depths, ($e^{-d} = 0$, $0{\cdot}1$ and $0{\cdot}2$). Part of the difficulty of interpretation is that the concept of group velocity is not readily extended to finite-amplitude waves. There is a substantial discussion of group velocities for finite-amplitude deep-water waves in Peregrine & Thomas (1979), § 6.

For waves of lesser deep-water steepness, say $\epsilon_\infty^2 \leqslant 0{\cdot}1$, linear theory does give a good indication of the initial changes, so that in figures 3(*b*) and 3(*c*) a large portion of the depth variation has been omitted.

In the solutions for waves of intermediate deep-water steepness, $0{\cdot}1 \geqslant \epsilon_\infty^2 \geqslant 0{\cdot}01$, it is very encouraging to see how the TSW approximation approaches Cokelet's solution and the calculated points then intertwine along the same line. For steep waves TSW is quite satisfactory for values of μ up to $0{\cdot}5$. We were unable to use Cokelet's method for long waves, as has been noted in § 2, and the points in figure 3(*b*) indicate where solutions were no longer obtainable. However, for these longer waves TSW is satisfactory if the amplitude is not too small. Cnoidal wave theory is not adequate until waves of lower deep-water steepness are considered. These are shown in figure 3(*c*).

For the waves shown in figure 3(*c*), no solutions with Cokelet's method are worth computing. Linear and cnoidal wave theory are quite satisfactory until the waves are of appreciable amplitude compared with the depth; in which case the TSW approximation can be used with confidence. Note, the discussion in § 6 indicates that these particular solutions can only rarely be directly useful.

Figure 4 shows the variation of wavenumber, k, set-down, $\delta = h-D$, and 'wave-induced return current', $-U$, for waves of deep-water steepness $a = 0{\cdot}371$, $\epsilon_\infty^2 = 0{\cdot}7$, $H/L = 0{\cdot}118$. The same quantities are shown in figure 5 for $a = 0{\cdot}0409$, $\epsilon_\infty^2 = 0{\cdot}01$, $H/L = 0{\cdot}013$. The variation of wavenumber shows no surprising features; at least, if one is already familiar with the two solutions for the steepest waves. In figure 5, which is where a greater variation is shown, as the amplitude increases so it can be seen that the wavelength is longer than linear theory predicts. Since the linear result comes directly from the dispersion relation one might expect that the nonlinear dispersion relation indicates the direction of departure from the linear result. For a constant frequency waves of greater steepness have a longer wavelength in a given depth of water and this is borne out by the results in figure 5. In figure 4 the linear and nonlinear results for k are so close because the wave-induced current causes a significant doppler shift, on the other hand the constant frequencies in the two solutions differ by about 7%. The wave set down is generally insignificant and does not exceed 1% of the local depth.



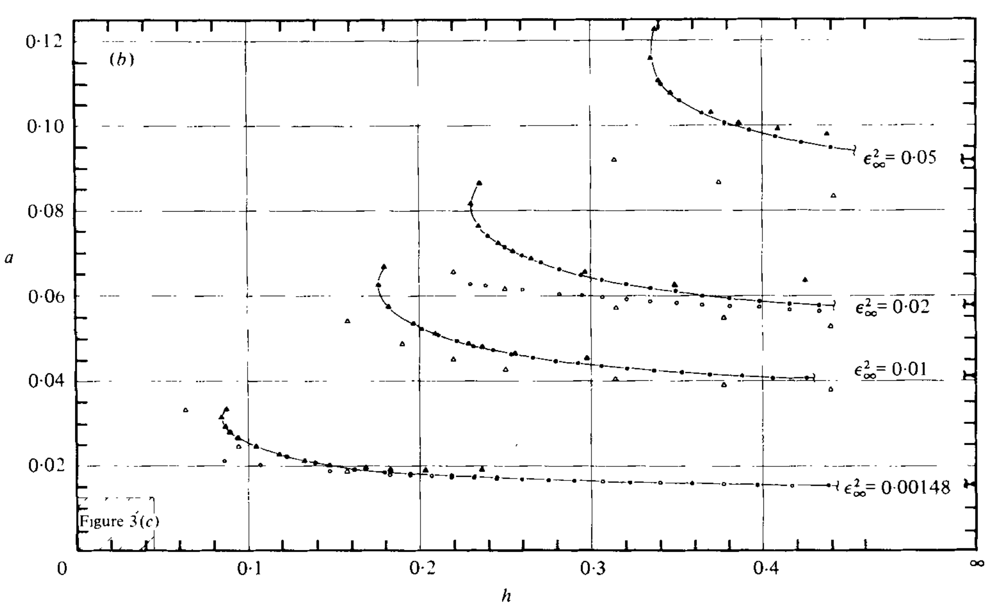


FIGURE 3 (*a*, *b*). For legend see facing page.

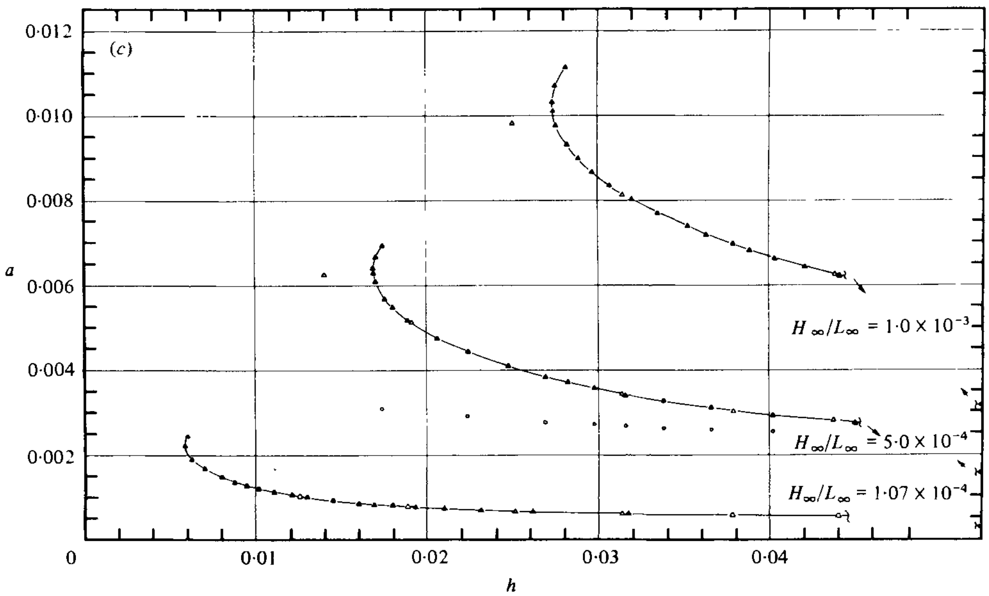


FIGURE 3. Variation of wave amplitude with still water depth. These are both made dimensionless with k_∞^*. Representative points for four theoretical solutions are shown. The line joins the most accurate solutions. (*a*), (*b*), (*c*) The points with a line through are less well determined than the others. ●, Cokelet's method; ▲, TSW; ○, linear theory; △, cnoidal theory.

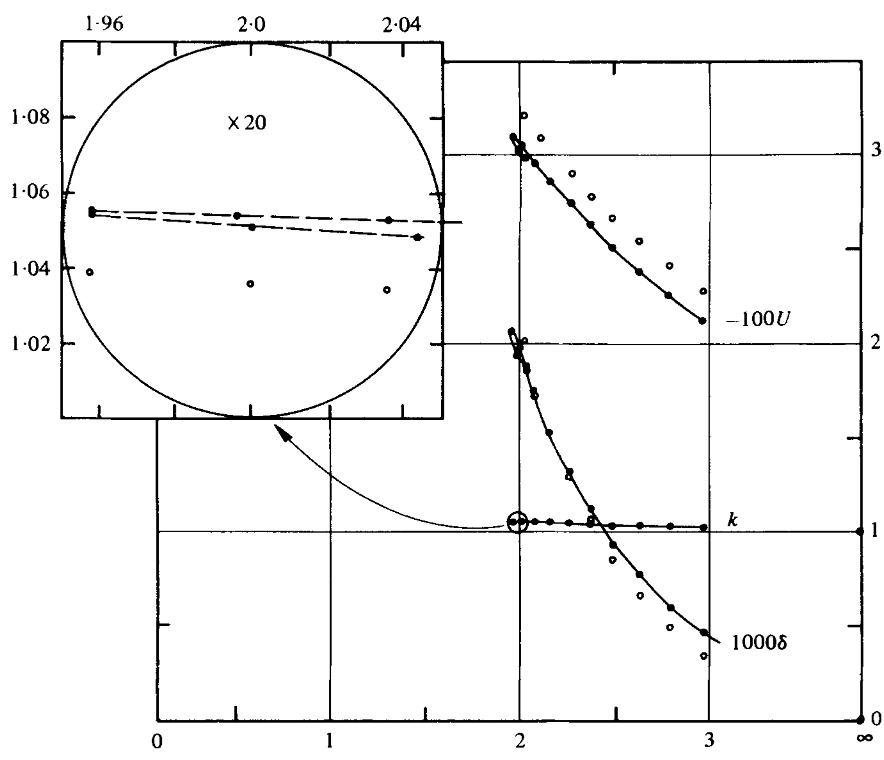


FIGURE 4. Theoretical variation of wavenumber k, set down δ, and current U, with depth for deep water waves with $\epsilon_\infty^2 = 0{\cdot}7$, $H_\infty/L_\infty = 0{\cdot}118$. ●—●, Cokelet's method; ○ ○, linear theory.

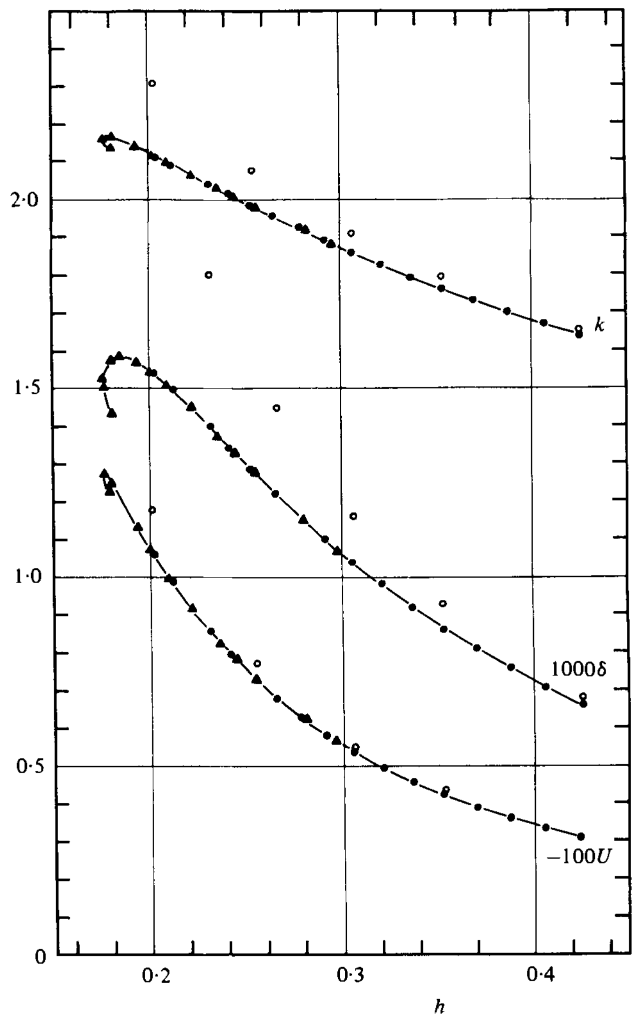


FIGURE 5. Theoretical variation of wavenumber k, set down δ, and current U, with depth for deep water wave with $\epsilon_{\infty}^2 = 0{\cdot}01$, $H_{\infty}/L_{\infty} = 0{\cdot}013$. ●, Cokelet's method; ▲, TSW; ○, linear theory.

5. Wave breaking

The results presented in this paper are based on the assumption that at any point the waves are locally similar to a train of periodic travelling waves. However, most waves incident on a beach break, and breaking is always asymmetrical about the wave crest, whereas the travelling wave solutions are all symmetrical. It is a reasonable conjecture that a small amount of asymmetry has little effect on the integral properties of waves. For example, both Yuen & Lake (1975, equation (5)) and Smith (1976, equation (4c)) have an asymmetrical term in their velocity potentials, owing to gradients of amplitude and current respectively, yet the resulting averaged equations, which are for 'faster' modulations than the equations used here, can be derived from properties of the usual symmetrical wave train. This suggests that a slowly-varying wave-train theory may have a greater applicability than appears at first sight.

The process of wave breaking is poorly understood. However, considerable progress has been made for waves on deep water. Longuet-Higgins (1978*a*, *b*) and Longuet-Higgins & Cokelet (1978) show that steep wave-trains in deep water have instabilities that lead to wave breaking. One instability is of a type that includes the modulational Benjamin–Feir instability. This is probably not very relevant in the context of waves on beaches since the growth of the instability is slow and it diminishes as the depth decreases. The other instabilities affect very steep waves and when they occur the waves break on a time scale of less than a wave period. These seem to be most relevant to wave breaking in this context.

One of these more rapid instabilities is associated with the maximum of the phase velocity as a function of steepness and the other appears to be associated with the maximum of the first Fourier component. There are corresponding maxima for waves on a finite depth of water, and it is a reasonable conjecture that these also are related to instabilities which rapidly lead to breaking.

The solutions presented in the previous section show singularities at steepnesses which are close to, though not precisely the same as, the steepness of maximum phase velocity. The singularities appear in the diagrams as vertical tangents to the solution curve. Clearly such an 'infinite' rate of change with depth contradicts the slowly-varying wave assumption. More significant is the absence of any solution for shallower depths and the large steepness at which the singularity appears. In the absence of any precise criterion for breaking it is reasonable to associate these singularities with a change in character of the waves which relatively rapidly leads to wave breaking. That is, we suggest that waves will break after travelling a distance of the order one wavelength past the point where the solution is singular.

The amplitude of waves at breaking may also be estimated from the amplitude of the singularity in the solution. However, since the details of wave breaking must depend on other influences, such as bottom slope; and the height of waves at breaking varies significantly with the way in which the wave breaks, e.g. a jet of water projected from the crest can rise a little at first, such a parameter is difficult to predict with our present incomplete knowledge of the breaking process.

By taking the 'breaking point' to be coincident with the singularity of a solution, we have plotted several wave parameters at breaking as functions of the deep-water wave steepness a_∞. These appear in figures 6 and 7 where all the curves have been extrapolated to a deep-water steepness of $a = 0{\cdot}41$. This corresponds to the lower boundary, in steepness, of the rapid instabilities found by Longuet-Higgins (1978*b*).

The variation of most properties shows a greater change from deep-water values as the initial deep-water steepness decreases. This is an effect of the greater-change of depth required before waves break. The left portion of each curve depends on the properties of steep solitary waves through the TSW approximation. For example, the amplitude/depth ratio asymptotes to just under half the height of the maximum solitary wave.

The variation of set-down, δ_b, and return current, $-U_b$, each show a maximum. In deep water there is no set-down; on the other hand, the wave-action flux for gentle waves is necessarily small thus again causing a small set-down. Hence, between the two extremes of waves which are close to breaking in deep water and very gentle deep-water waves a maximum is to be expected.

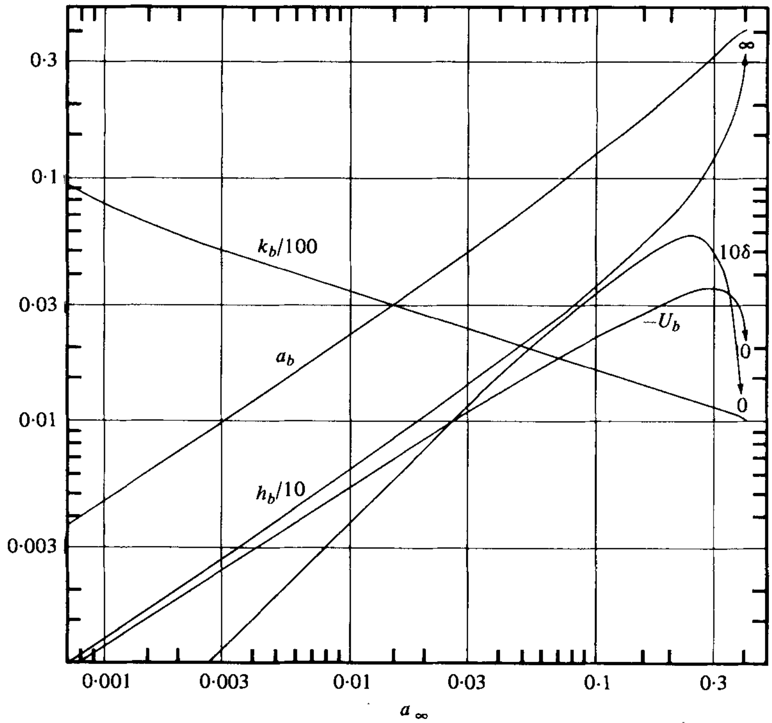


FIGURE 6. Theoretical variation at the 'breaking point' of wavenumber k_b, amplitude a_b, depth h_b, set down δ_b, and current U_b, with deep-water steepness $a^*_\infty k^*_\infty = a_\infty$.

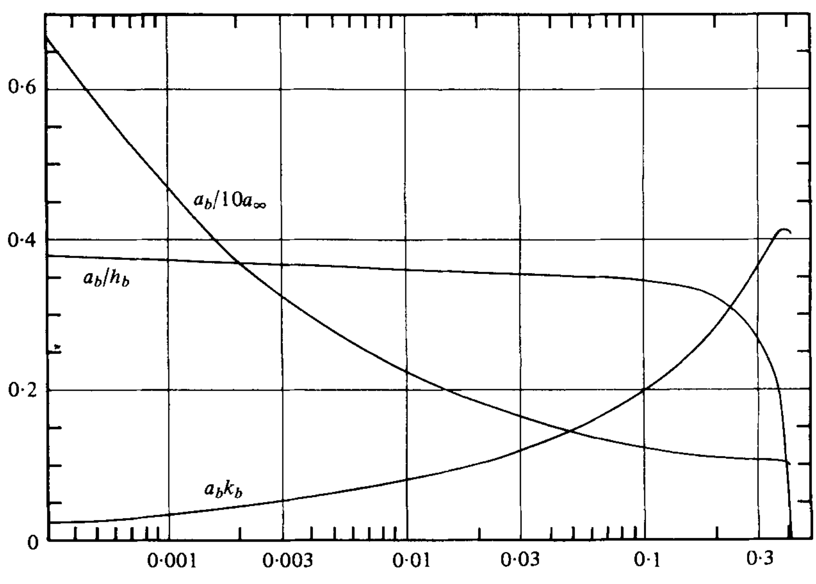


FIGURE 7. Theoretical variation at the 'breaking point' of amplitude amplification a_b/a_∞, amplitude to depth ratio a_b/h_b, and wave steepness $a_b k_b$, with deep-water steepness $a^*_\infty k^*_\infty = a_\infty$.

6. Limitations of the theory

The basic solutions used here give accurate values for the properties of trains of periodic surface waves on water. All previous theoretical descriptions of this problem have had to use approximate solutions for the periodic waves or other approximations to the mathematical problem. It is thus worth examining carefully the remaining sources of error.

There are the inaccuracies due to the mathematical idealization of the physical situation. That is the motion is assumed to be inviscid and irrotational. In all real examples of water waves there is dissipation and non-uniform currents are generated. Dissipation could be readily incorporated into this mathematical approach if only appropriate expressions for bottom stress and energy dissipation were available for *all* water waves. Some expressions which might be used include the laminar dissipation of cnoidal waves (Miles 1976) and empirical results for waves with turbulent boundary layers by Kamphuis (1975) and by Jonsson & Carlsen (1976).

Even if wave trains enter a region without currents, they soon generate currents, even in a laboratory wave flume. These are not readily predicted, e.g. see Dore (1977) for some recent work, and may not even be steady for steady wave conditions (Russell & Osorio 1958). Even if these rotational currents are known it is not a trivial matter to include their effects on the waves, e.g. see the review by Peregrine (1976). However, their order of magnitude is usually small, of the order 1% of the wave's phase velocity (e.g. compare U_b and k_b^{-1} from figure 6).

The *assumptions* required for the mathematical solution imply the following wave properties. (i) Wave reflexion is insignificant, (ii) the waves are periodic, and (iii) the rate of change of depth is sufficiently small that waves can 'adjust' to changes in depth in a manner which is not sensitive either to the rate of change of slope or the direction of slope of the bed (note: we are assuming that waves retain their symmetry of shape about their crests). These first two properties are unimportant in the linear theory for which wave solutions are superposed without interaction. For non-linear waves this is not so. However for gentle beaches where there is little reflexion one should not expect reflexion to modify the local amplitudes by any more than the amplitude of the reflected wave.

Precise periodicity of incident waves is difficult to achieve even in a laboratory experiment. In nature, waves from deep water do not have constant amplitude even when the spectrum is very narrow, as with distant swell. Locally generated waves have a broader spectrum and there may well be important nonlinear effects in this case; there is certainly a wide spread of positions at which individual waves crests break. For the present we cannot expect theory to be very helpful in the latter case, but some progress may be made with near-linear theory for the former example by considering modulation equations (e.g. see Lake & Yuen 1978).

The third property, that the wave train 'adjusts' to changes in depth has only been considered briefly in the past. For example, Svendsen & Hansen (1976) consider that the appropriate parameter to be kept small is the relative change of depth over one wavelength, i.e.

$$S = \alpha L/h \ll 1, \tag{21}$$

where α is the bed slope $|dh/dx|$. The matter is discussed again in Svendsen & Hansen

(1978) for the Boussinesq equations. A non-zero bed slope does lead to an asymmetry of waves about their crests, and in this second paper, it is calculated for cnoidal waves and a satisfactory comparison with experiment is achieved. Since this asymmetry is travelling with the wave, it is possible to calculate its effect on the integral properties of the wave train and to refine the slowly-varying wave theory for that solution. We have not done this; the result would depend on the local bed slope. However, an order of magnitude estimate is possible from Svendsen & Hansen's (1978) solution. This indicates a proportionate correction to the energy flux of $(L/h)^3 \alpha G(Ur)$ where $G(Ur)$ is a function of the Ursell number

$$Ur = HL^2/h^3. \tag{22}$$

The solution curves for G show that it is $O(10^{-4})$ for a wide range of Ur so that we need

$$\alpha \ll 10^{-4}(h/L)^3. \tag{23}$$

This last result suggests that it is the longer waves which are least likely to adjust to changes in depth. Some idea of why this is so can be obtained by considering the two extremes. In deep water, the group velocity is only one half the phase velocity and this is a clear indication that any one wavelength of a wave train is interacting strongly with its environment (i.e. the other waves of the train). This strong interaction can lead to relatively rapid changes in the properties of a single wave crest.

On the other hand, the longest waves are described well by a train of solitary waves. Each wave crest is separated by a flat trough from its neighbours and is almost independent of them. Thus in many circumstances the crests can be expected to respond to changes of depth individually. For example, Miles (1979) gives a near-linear theoretical discussion of a solitary wave on a beach. It is shown how mass in the wave is not conserved, some of the wave trails behind the crest and for a consistent theory the reflected wave must be included. Numerical solutions are presented in Peregrine (1967).

In the next section a quantitative assessment of the interaction between solitary waves is used to find a criterion for judging when a beach is of sufficiently mild slope.

7. Interactions between wave crests in shallow water

A measure of the interaction between solitary-wave crests can be obtained by considering just two waves. There is no solution corresponding to two equal solitary waves propagating unchanged. They do interact. Solutions which do have two equal solitary waves at one instant are solutions that describe the interaction between two solitary waves which commence with slightly differing amplitudes. The larger wave catches up with the smaller, they interact and exchange identities. At the mid-point of the interaction the wave crests have equal amplitudes.

The solution of the Korteweg–de Vries equation for two solitary waves is given by Whitham (1974, equation (17.21)). A space–time sketch of the wave crest trajectories is given in figure 8. It is difficult to use the explicit solution to find the minimum distance between the crests in terms of the original difference in amplitude. However, it is straightforward to obtain an approximate value by using their change of phase. When this is transformed to the physical (x^*, t^*) plane from Whitham's dimensionless variables, the following is obtained.

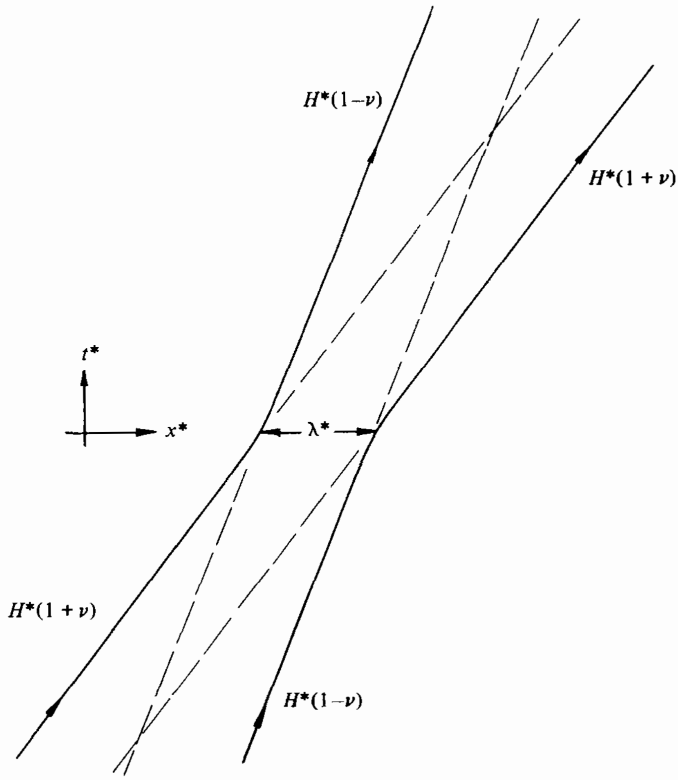


FIGURE 8. The trajectories of two almost equal solitary waves in space–time.

Two solitary waves of heights $H^*(1+\nu)$ and $H^*(1-\nu)$ on depth h^* of water have a minimum separation λ^* given by

$$\lambda^*/h^* \simeq 2(h^*/3H^*)^{\frac{1}{2}} \log(2/\nu), \tag{24}$$

where $\nu \ll 1$.

A good measure of the rate of interaction between the crests is the rate of change of crest height at the mid-point of the interaction, say $d\eta^*/dt^*$. Again this is difficult to determine from the explicit solution, but Lax (1968) derives an ordinary differential equation for a corresponding dimensionless quantity, m_t, in his equation (2.28). After some algebra this yields:

$$\frac{d\eta^*}{dt^*} = \frac{\nu^2(3gh^*)^{\frac{1}{2}}}{2}\left(\frac{H^*}{h^*}\right)^{\frac{5}{2}}. \tag{25}$$

Elimination of ν between (25) and (24) yields

$$\frac{d\eta^*}{dt^*} = 2(3gh^*)^{\frac{1}{2}}\left(\frac{H^*}{h^*}\right)^{\frac{5}{2}} \exp\left[-\frac{\lambda^*}{2h^*}\left(\frac{3H^*}{h^*}\right)^{\frac{1}{2}}\right], \tag{26}$$

as a measure of the interaction of wave crests of height H^* and distance λ^* apart.

Now, we need to compare this with the interaction of a wave crest with the bottom slope. If we follow Miles (1979), the major variation of amplitude of a solitary wave is given by his equation (4·2), which can be written as

$$\eta^* h^* = \text{constant}.$$

Thus the rate of change of wave height due to interaction with the bottom is

$$\frac{d\eta^*}{dt^*} = \frac{\eta^*}{h^*}\left(-\frac{dh^*}{dx^*}\right)\frac{dx^*}{dt^*} \simeq \frac{H^*}{h^*}\alpha(gh^*)^{\frac{1}{2}}. \tag{27}$$

In a periodic wave train in water of constant depth the interactions between adjacent (and possibly more distant) crests balance in such a way that the wave profile is in steady translation. In water of slowly-varying depth, when there is sufficient distance between crests that the TSW approximation is accurate, it is quite possible for each crest to behave like an individual solitary wave. (In extreme cases, individual incident waves disintegrate into a succession of secondary crests.) Thus, the waves can be correctly represented as a period wave train only if the interaction between crests, (26), is more rapid than the interaction with the slope (27). After a little rearrangement we obtain:

$$a < 2(3)^{\frac{1}{2}}(H^*/h^*)^{\frac{3}{2}} \exp\left[-\tfrac{1}{2}(3Ur)^{\frac{1}{2}}\right], \tag{28}$$

where Ur is the Ursell number defined earlier in equation (22).

It is well known the Ursell number is large for long waves such as we are considering. Thus the exponential in (28) is small. The factor multiplying the exponential is at most $O(1)$ and small when H^*/h^* is small. The exponential has the values 0·01, 0·001, 0·0001 at values of Ur of 28, 64, and 113 respectively. In this context it is more useful to note that the TSW approximation becomes accurate in figure 3 when Ur is in the range 80 to 100. The corresponding values of the exponential in expression (28) are $4{\cdot}3\times10^{-4}$ to $1{\cdot}7\times10^{-4}$. This implies the TSW solutions are only appropriate for beach slopes of the order 10^{-4}. This is rather a severe restriction.

8. Comparison with experiment

A substantial amount of experimental data has been measured for waves on beaches. We make our comparison with the recent data of Hansen & Svendsen (1979). A wide range of incident waves were measured by them for the single beach slope of 1 in 35. Advantages of these experiments are that considerable care is taken to generate periodic waves and minimize free harmonics (see Hansen & Svendsen 1974); and measurements of height, wave velocity and wave profiles are given.

Three particular experiments have been selected for comparison almost spanning the range of periods and steepnesses used. In figures 9(*a*), 10 and 11 the experimental measurements of the wave height are shown together with three, theoretical curves. The agreement between them is about as good as can be expected without the inclusion of some dissipation in the theory. We have refrained from including dissipation since there is no reliable method of estimating it for the highest waves which are of the greatest interest in this work. Comparisons with Svendsen & Hansen's (1977) work which shows experiments together with cnoidal wave curves which include dissipation indicate that at least for the lower waves this would lead to good agreement. In each diagram there is an indication of the length of the highest calculated waves. In every case the experimental waves have broken within half a wavelength of the cessation of solutions. This is in tune with the discussion of § 5. The wave height of the 'breaking point' solution is also close to the maximum recorded wave height. The breaking amplitude is not indicated in the experimental records but can reasonably be expected to be close to the maximum.

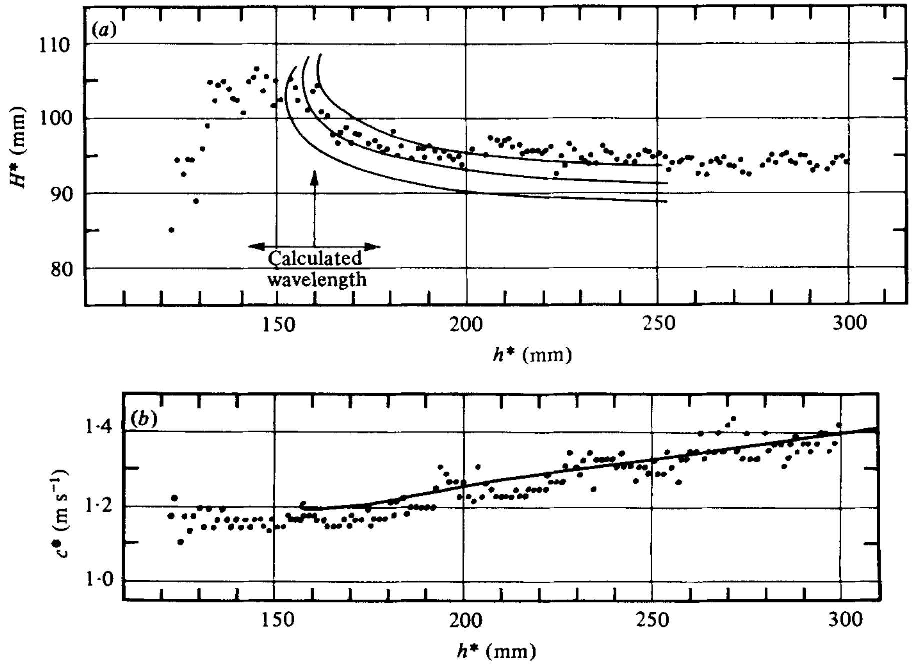


FIGURE 9. Comparison of theory and experiment. (*a*) Wave height against water depth, the theoretical wavelength of the limiting wave is indicated. (*b*) Wave phase velocity against water depth. Test 101101. Mean period 1·0 s; theoretical deepwater values for the middle curve are H^*_∞ = 96 mm, $a^*_\infty k^*_\infty$ = 0·192, ϵ^2_∞ = 0·21.

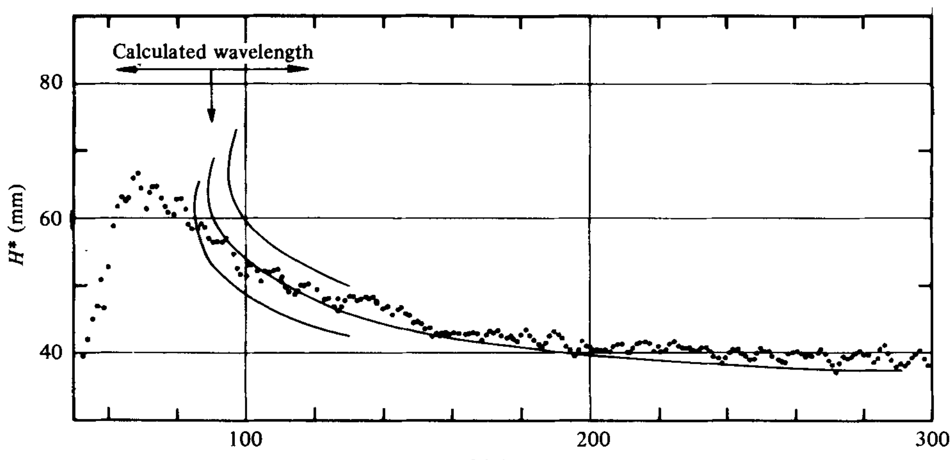


FIGURE 10. Comparison of theory and experiment; wave height against water depth, the theoretical wavelength of the limiting wave is indicated. Test A06103. Mean period 1·67 s; theoretical deep water values for the middle curve are H^*_∞ = 37·5 mm, $a^*_\infty k^*_\infty$ = 0·027, ϵ^2_∞ = 0·0044.

Figure 9(*b*) shows a comparison between theory and experiment for the waves' phase velocity. Only one theoretical curve is shown since the other two of figure 9(*a*) are very close.

The wave profiles near breaking were measured by Hansen & Svendsen and are shown for these three wave conditions in figure 12. They are measured as profiles in

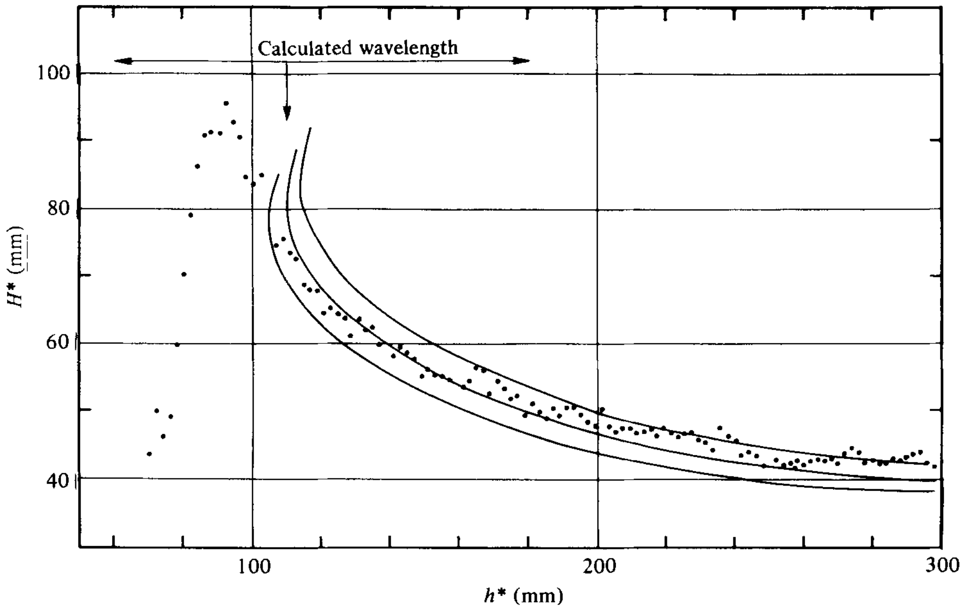


FIGURE 11. Comparison of theory and experiment: wave height against water depth, the theoretical wavelength of the limiting wave is indicated. Test 031041. Mean period 3·33 s; theoretical deep water values for the middle curve H^*_∞ = 30·5 mm, $a^*_\infty k^*_\infty$ = 0·0053.

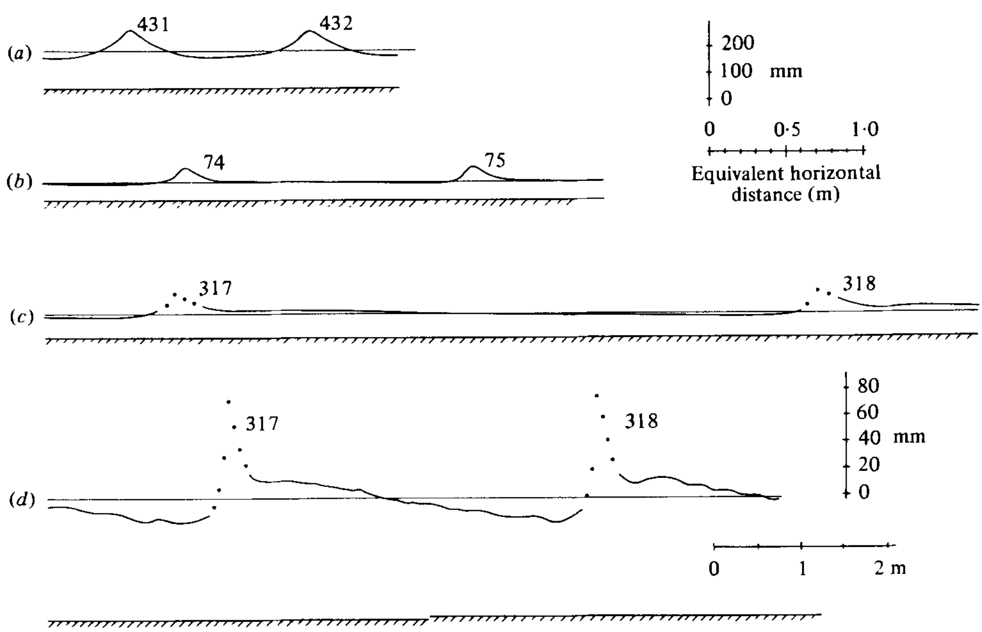


FIGURE 12. Profiles of waves in Hansen & Svendsen's (1979) experiments used for comparison in figures 9–11. They are all for the highest, or close to highest, waves. The profiles were measured as a function of time from a slowly-moving carriage. This has been converted to an equivalent horizontal distance, using the wave velocity. The vertical exaggeration is 7:4 for figures (*a*), (*b*) and (*c*), and is 15:1 for figure (*d*). (*a*) Test 101101: wave 430, D^* = 147·6 mm; wave 431, D^* = 146·4 mm. (*b*) Test A06113 (identical conditions to test A06103): wave 74, D^* = 71·0 mm; wave 75, D^* = 69·8 mm. (*c*) and (*d*) Test 031041; wave 317, D^* = 94·7 mm; wave 318, D^* = 92·7 mm. Individual data points are shown in the places where a smooth profile cannot easily be drawn.

time but are drawn approximately to the same horizontal equivalent length-scale. The longest waves have too few data points to define the crests well and individual data points are shown in the curve. The shortest wave is reasonably symmetrical, asymmetry of the intermediate wave is noticeable but not strong. The longest wave is very strongly asymmetrical. It is also plotted with a greater vertical exaggeration since this shows its features more clearly. Each crest is developing strong features of its own, there is a 'shelf' behind each wave, due to the lack of mass conservation in the slowly-varying solitary-wave crest. In the final wave (number 318) there is a clear secondary crest. It is surprising that the comparison with the theory is so good, its Ursell number is about 3000. Even for the shortest wave in figure 12 *Ur* is quite large, approximately 80. This does raise a question as to whether the departure between theory and experiment is due entirely to dissipation since individual solitary wave crests would tend to gain amplitude slower than the theoretical solutions shown. The behaviour of individual crests may become the more important factor as waves approach breaking.

9. Conclusion

The effect of gradual variations of water depth on a periodic wave train have been investigated by using accurate solutions for periodic travelling waves on water of constant depth. It proved necessary to use an approximation of a train of solitary waves to accurately describe long waves.

The solutions show that there is a limiting depth for every wave train as it propagates into shoaling water. The relevance of the limiting solution to wave breaking is discussed and comparison with experiment indicates that there is as good agreement as can be expected. The comparison with experiment also shows that dissipation is significant, but it is possible that other effects are of comparable significance.

These other effects are described in a discussion of the limitations of the theory. In particular a criterion for its application to long waves is deduced, and from that it is concluded that when the TSW solution is accurate, the method is only appropriate for beach slopes of order 10^{-4}.

Anyone wishing to use this method as a working tool is advised to consider revising the method used to solve for the high-order Stokes'-wave coefficients. Longuet-Higgins (1978*c*) gives quadratic relations between them which can replace the cubic relations used by Cokelet.

The method can be extended to a wider class of problems. Work is in progress on the closely related problem of waves incident at an angle to a beach with straight, parallel contours, and further work is envisaged to investigate waves in the neighbourhood of caustics. A more challenging problem is to deal with problems of more general topography, that is problems equivalent to ray-tracing for linear waves.

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